Universal quantum computation with two-level trapped ions

Andrew M. Childs and Isaac L. Chuang
IBM Almaden Research Center, 650 Harry Road, San Jose, CA 95120
(14 August 2000)

Although the initial proposal for ion trap quantum computation made use of an auxiliary internal level to perform logic between ions, this resource is not necessary in principle. Instead, one may perform such operations directly using sideband laser pulses, operating with an arbitrary (sufficiently small) Lamb-Dicke parameter. We explore the potential of this technique, showing how to perform logical operations between the internal state of an ion and the collective motional state and giving explicit constructions for a controlled-NOT gate between ions.

I. INTRODUCTION

Ion trap quantum computation, first introduced by Cirac and Zoller [1], is a potentially powerful technique for the storage and manipulation of quantum information [1–4]. In this scheme, information is stored in the spin states of an array of trapped ions and manipulated using laser pulses. Reasonably long coherence times can be achieved, compared to achievable switching rates [5], and individual qubits can be addressed through spatial separation of the ions. Experimental implementations of this scheme have succeeded in performing simple two-qubit logic gates [6,7] and preparing entangled states [8,9].

An ion trap quantum computer may be modeled as a collection of N particles with spin $\frac{1}{2}$ in a one-dimensional harmonic potential. Laser pulses incident on the ions can be tuned to simultaneously cause internal spin transitions and vibrational (phonon) excitations, thus allowing local internal states to be mapped into shared phonon states. In this manner, quantum information can be communicated between any pair of ions and logic gates can be performed.

In this paper, we consider an interesting question which arises in this scenario: what is the simplest internal spin state structure required by each ion? In the original Cirac-Zoller formulation, ions with three levels are required. However, as the Cirac-Zoller method is generalized to other physical systems, such as neutral trapped atoms [10] or quantum dots in an electromagnetic cavity [11], it has become highly desirable to determine whether just *two* internal levels are sufficient for performing universal quantum computation.

Previously, Monroe *et al.* have shown how a controlled-NOT can be performed between an ion and a phonon state using only two-level ions [12]. Their method depends on fine-tuning of the Lamb-Dicke parameter, which relates the laser frequncy to the scale of the ions' wave functions, to cancel unwanted side effects. Furthermore, they do not allow the Lamb-Dicke parameter to be arbitrarily small; the lowest value quoted in [12] is 0.316.

Here, we provide a general and accessible technique for performing universal logic between ions with only two internal levels. This scheme operates with any sufficiently small value of the Lamb-Dicke parameter, and also introduces new ways to utilize (or avoid) particular phonon states while performing quantum logic gates. We begin in Section II by presenting the allowed operations using the usual Jaynes-Cummings model for spin-boson interactions. We then describe a multiple-pulse construction for the controlled-NOT gate in Section III and give further constructions in Section IV before concluding with some possible extensions of the work.

II. THEORETICAL ION TRAP MODEL

The energy level diagram of one ion, including the motional state, is shown in Fig. 1. For the sake of definiteness, we may think of the motional levels as corresponding to the center of mass degree of freedom. However, they could just as well correspond to another mode of oscillation, such as the "breathing" mode of a pair of ions. In practice, one might wish to choose a mode other than the center of mass to achieve reduced susceptibility to decoherence [13].

The Hamiltonian of this system is $H_0 = \hbar\omega_0\frac{\sigma_z}{2} + \hbar\omega_z a^\dagger a$, where σ_z is a Pauli spin operator for the nuclear spin and a annihilates a phonon. Throughout this paper, we work in the frame of this Hamiltonian. Turning on the electromagnetic field of a laser gives an interaction Hamiltonian

$$H_I = -\vec{\mu}\cdot\vec{B}\,, \tag{1}$$

where $\vec{\mu} = \mu\vec{\sigma}/2$ is the magnetic moment of the ion and $\vec{B} = B\hat{x}\cos(kz-\omega t+\Phi)$ is the magnetic field produced by the laser. Here $z = z_0(a+a^\dagger)$, where $z_0 = \sqrt{\hbar/2Nm\omega_z}$ is a characteristic length scale for the motional wave functions and m is the mass of an ion.

We consider the regime in which $\eta \equiv kz_0 \ll 1$. In this regime, we may determine the effect of a laser pulse at a specific frequency ω by expanding Eq. (1) in powers of η and neglecting rapidly rotating terms. Pulsing on resonance ($\omega = \omega_0$) allows one to perform the transformation

$$R(\theta,\phi) = \exp\left[i\frac{\theta}{2}(e^{i\phi}\sigma^+ + e^{-i\phi}\sigma^-)\right] , \qquad (2)$$

allowing one to do arbitrary single-qubit operations on an ion's internal state. Pulsing at the nth blue sideband frequency ($\omega = \omega_0 + n\omega_z$) gives [14]

$$R_n^+(\theta,\phi) = \exp\left[i\frac{\theta}{2}(e^{i\phi}\sigma^+ a^n + e^{-i\phi}\sigma^- a^{\dagger n})\right] \qquad (3)$$

and pulsing at the nth red sideband frequency ($\omega = \omega_0 - n\omega_z$) gives

$$R_n^-(\theta,\phi) = \exp\left[i\frac{\theta}{2}(e^{i\phi}\sigma^+ a^{\dagger n} + e^{-i\phi}\sigma^- a^n)\right] . \qquad (4)$$

Here, $\sigma^\pm = (\sigma_x \pm i\sigma_y)/2$ act on the internal state of the ion. In each case, the parameter θ depends on the strength and duration of the pulse and ϕ depends on its phase. For an nth order transition of duration t, θ is given by

$$\theta = -\frac{\mu B t \eta^n}{2\hbar n!} . \qquad (5)$$

Also,

$$\phi = \Phi + (n \bmod 4)\frac{\pi}{2} . \qquad (6)$$

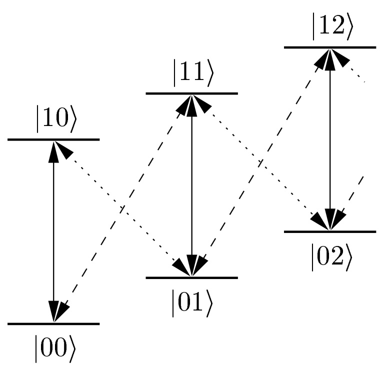

FIG. 1. Energy level diagram for a single ion's nuclear spin state along with the motional mode, showing only the lowest three motional states. We denote the state of a particular ion using the ket $|pq\rangle$, where $p \in \{0,1\}$ is the nuclear spin state and $q \in \{0,1,2,\ldots\}$ is the motional state. Transitions at the on-resonance frequency ω_0 are shown with solid lines. The first blue ($\omega_0 + \omega_z$) and red ($\omega_0 - \omega_z$) sideband transitions are shown using dashed and dotted lines, respectively. Higher-order transitions are suppressed for clarity.

III. CNOT USING THE FIRST MOTIONAL SIDEBAND

We suppose that the state of the ion trap quantum computer begins in the motional $|0\rangle$ state. We may excite higher-order motional levels using $R_n^\pm$ so long as we return to the motional ground state at the end of the gate. It is simplest to consider cases where only a few higher-order levels are relevant. For a particular ion, we refer to the subspace spanned by $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ (using the notation of Figure 1) as the *computational subspace* (CS).

First, we note that for the special values $\theta = j\pi\sqrt{2}$, where j is an integer, $R_1^\pm$ preserve the CS — that is, they map states within the CS to other states within the CS. For example,

$$R_1^+(\pi\sqrt{2}, -\pi/2) = \begin{pmatrix} \cos\frac{\pi}{\sqrt{2}} & 0 & 0 & \sin\frac{\pi}{\sqrt{2}} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\frac{\pi}{\sqrt{2}} & 0 & 0 & \cos\frac{\pi}{\sqrt{2}} \end{pmatrix} . \qquad (7)$$

Next, note that it is easy to produce another gate which preserves the CS by conjugation. For example, consider the gate $G = G_{\rm in} \otimes G_{\rm out}$, where $G_{\rm in}$ has support only on $\{|00\rangle, |10\rangle, |11\rangle\}$ and $G_{\rm out}$ acts on the rest of the space. Since R_1^+ does not connect these subspaces, conjugating this gate by R_1^+ produces another gate which preserves the CS. In other words,

$$R_1^+(-\theta,\phi) G R_1^+(\theta,\phi) \qquad (8)$$

preserves the CS. Using this observation, we can easily diagonalize Eq. (7), giving the gate

$$\begin{aligned} P &= R_1^+(-\pi/2, 0) R_1^+(\pi\sqrt{2}, -\pi/2) R_1^+(\pi/2, 0) \\ &= \mathrm{diag}(e^{i\pi/\sqrt{2}}, -1, 1, e^{-i\pi/\sqrt{2}}) . \end{aligned} \qquad (9)$$

This gate uses first-order sideband pulses for a total duration of $(1+\sqrt{2})\pi$, only slightly longer than the 2π duration required for the equivalent step in the Cirac-Zoller scheme.

With this diagonal gate and R_1^-, it is straightforward to construct a controlled-NOT gate between two ions using a sequence similar to the original Cirac-Zoller construction. Note that $R_1^-(\pi, -\pi/2)$ allows us to interchange $|01\rangle$ and $|10\rangle$ (up to a phase), performing

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad (10)$$

on the subspace $\{|00\rangle, |01\rangle, |10\rangle\}$. Assuming the motional state is initially $|0\rangle$, this corresponds to a swap between the state of the ion and the motional state. In fact, we can use a version of this gate with arbitrary phase to do logic between ions. Thus we can perform a controlled-NOT from ion j (the control) to ion k (the target) using

$$\begin{aligned} CNOT_{jk} = {} & Z_j(-\pi/(2\sqrt{2})) R_{1j}^-(\pi,\phi) \qquad (11) \\ & H_k P_k Z_k(-\pi/(2\sqrt{2})) H_k R_{1j}^-(\pi,\phi) , \end{aligned}$$

for any value of ϕ, where a subscript denotes which ion is acted on and we have introduced the single-qubit gates

$$Z(\phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} , \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} . \qquad (12)$$

IV. OTHER GATE CONSTRUCTIONS

Although the construction by which we arrived at Eq. (9) is relatively straightforward, there are other ways to construct gates which preserve the CS. In fact, it is possible to perform a generalization of Eq. (9) by considering the form

$$\begin{aligned} U(\phi) = R_1^+(-\alpha,\pi/2)R_1^+(-\beta,\gamma)R_1^+(2\phi,\delta) \\ R_1^+(\beta,\gamma)R_1^+(\alpha,\pi/2)\,. \end{aligned} \tag{13}$$

Making the choices

$$\cos\alpha = \cos\sqrt{2}\alpha \tag{14}$$
$$\cos\beta = \cos\sqrt{2}\beta \tag{15}$$
$$\mathrm{sign}(\sin\alpha)\mathrm{sign}(\sin\beta) = \mathrm{sign}(\sin\sqrt{2}\alpha)\mathrm{sign}(\sin\sqrt{2}\beta) \tag{16}$$
$$\delta - \gamma = \sin^{-1}\left(\frac{\cos\alpha}{\sin\beta}\right) \tag{17}$$
$$\gamma = \tan^{-1}[-\cos\beta\tan(\delta-\gamma)] \tag{18}$$

results in

$$U(\phi) = \mathrm{diag}(e^{i\phi}, e^{\pm i\sqrt{2}\phi}, 1, e^{-i\phi})\,. \tag{19}$$

For example, there is such a solution with $\alpha \approx 298.2°$, $\beta \approx 149.1°$, $\gamma \approx 63.87°$, and $\delta \approx 131.0°$.

The source of the factors of $\sqrt{2}$ in gates derived from the first-order sideband is the matrix element $\langle 1|a|2\rangle = \sqrt{2}$ for harmonic oscillator states. However, note that

$$\langle 1|a^3|4\rangle = 2\langle 0|a^3|3\rangle = 2\sqrt{6}\,. \tag{20}$$

The integer ratio of these matrix elements suggests that third-order transitions may be used to create simpler diagonal gates. Indeed, we find

$$R_3^-(2\pi/\sqrt{6},\phi') = \mathrm{diag}(1,1,-1,1) \tag{21}$$

for any value of ϕ'. This differs from a controlled-Z gate by only single-qubit operations, so we can simply use the Cirac-Zoller construction to produce a CNOT between ions:

$$\begin{aligned} CNOT_{jk} = H_k Z_j(\pi/2)R_{1j}^-(\pi,\phi)R_{3k}^-(2\pi/\sqrt{6},\phi') \\ Z_k(-\pi/2)R_{1j}^-(\pi,\phi)H_k \end{aligned} \tag{22}$$

Here, we require fewer pulses than in Eq. (11). However, note that the third-order sideband pulse must be longer by a factor of order η^{-2} than the first-order sideband pulses for the same laser intensity.

Most of these results generalize to other choices of the computational subspace. For example, consider using $\{|00\rangle, |02\rangle, |10\rangle, |12\rangle\}$. Analogous to Eq. (11), we find

$$\begin{aligned} CNOT_{jk} = Z_j(-\pi/(2\sqrt{6}))R_{2j}^-(\pi/\sqrt{2},\phi)H_k \\ R_{2k}^+(-\pi/(2\sqrt{2}),0)R_{2k}^+(\pi/\sqrt{3},-\pi/2) \\ R_{2k}^+(\pi/(2\sqrt{2}),0)Z_k(-\pi/(2\sqrt{6})) \\ H_k R_{2j}^-(\pi/\sqrt{2},\phi)\,. \end{aligned} \tag{23}$$

Similarly, analogous to Eq. (22) (exploiting the integral relationship $\langle 2|a^7|9\rangle = 6\langle 0|a^7|7\rangle$),

$$\begin{aligned} CNOT_{jk} = H_k Z_j(\pi/2)R_{2j}^-(\pi/\sqrt{2},\phi) \\ R_{7k}^-(\pi/(6\sqrt{35}),\phi')Z_k(-\pi/2) \\ R_{2j}^-(\pi/\sqrt{2},\phi)H_k\,. \end{aligned} \tag{24}$$

Finally, we wish to point out that it is conceivable to treat the lowest two motional levels as an additional qubit, rather than simply an intermediary for logic, if one can perform a true swap operation over the entire computational subspace. Note that

$$\begin{aligned} &R_1^-(2l\pi\sqrt{2},-\pi/2) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(l\pi\sqrt{2}) & \sin(l\pi\sqrt{2}) & 0 \\ 0 & -\sin(l\pi\sqrt{2}) & \cos(l\pi\sqrt{2}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{25}$$

We may get arbitrarily close to a swap operation by choosing large enough l with $\cos(l\pi\sqrt{2}) \approx 0$. For example, choosing $l = 2378$ gives a swap operation to a precision of about 10^{-3}. Although this observation may not be useful in practice, it shows that universal quantum logic including a motional qubit is not forbidden in principle.

V. CONCLUSIONS

We have demonstrated the possibility of doing ion trap quantum computation with two-level ions, allowing any sufficiently small value of the Lamb-Dicke parameter. While we believe that a construction similar to Eq. (11) will be most useful in practice, we have shown that there are many other ways to realize a controlled-NOT gate between ions without using extra levels, some of which might be useful given appropriate experimental conditions.

Our initial motivation for gates such as Eqs. (9) and (19) came from the theory of composite pulses used in the art of NMR, in which errors in pulse length or frequency are canceled at low order using a sequence of pulses. Thus one may accomplish with a composite pulse, in the presence of errors, what a single pulse would have performed in the absence of errors [15]. In the present case, the goal is somewhat different: we construct a sequence of gates which take the system into higher motional levels and back again to perform some logical operation. Nevertheless, the idea of stringing together several pulses to perform a simple gate has proved fruitful. It is our hope that further progress can be made in the experimental implementation of quantum information processing devices by considering how existing tools can be applied to different physical implementations.

[1] J. I. Cirac and P. Zoller, *Quantum computations with cold trapped ions*, Phys. Rev. Lett. **74**, 4091 (1995).

[2] D. J. Wineland, C. Monroe, W. M. Itano, D. Leibfried, B. E. King, and D. M. Meekhof, *Experimental issues in coherent quantum-state manipulation of trapped atomic ions*, J. Res. Natl. Inst. Stand. Tech. **103**, 259 (1998).

[3] A. Steane, *The ion trap quantum information processor*, App. Phys. B **64**, 623 (1997).

[4] R. J. Hughes, D. F. V. James, J. J. Gomez, M. S. Gulley, M. H. Holzscheiter, P. G. Kwiat, S. K. Lamoreaux, C. G. Peterson, V. D. Sandberg, M. M. Schamer, C. M. Simmons, C. E. Thornburn, D. Typa, P. Z. Wang, and A. G. White, *The Los Alamos trapped ion quantum computer experiment*, Fortsch. Phys. **46**, 329 (1998).

[5] D. P. DiVincenzo, *Quantum computation*, Science **270**, 255 (1995).

[6] C. Monroe, D. M. Meekhof, B. E. King, W. M. Itano, and D. J. Wineland, *Demonstration of a fundamental quantum logic gate*, Phys. Rev. Lett. **75**, 4714 (1995).

[7] Ch. Roos, Th. Zeiger, H. Rohde, H. C. Nägerl, J. Eschner, D. Leibfried, F. Schmidt-Kaler, and R. Blatt, *Quantum state engineering on an optical transition and decoherence in a Paul trap*, Phys. Rev. Lett. **83**, 4713 (1999).

[8] Q. A. Turchette, C. S. Wood, B. E. King, C. J. Myatt, D. Leibfried, W. M. Itano, and C. Monroe, *Deterministic entanglement of two trapped ions*, Phys. Rev. Lett. **81**, 3631 (1998).

[9] C. A. Sackett, D. Klelpinski, B. E. King, C. Langer, V. Meyer, C. J. Myatt, M. Rowe, Q. A. Turchette, W. M. Itano, D. J. Wineland, and C. Monroe, *Experimental entanglement of four particles*, Nature **404**, 256 (2000).

[10] T. Pellizzari, S. A. Gardiner, J. I. Cirac, and P. Zoller, *Decoherence, continuous observation, and quantum computing: A cavity QED model*, Phys. Rev. Lett. **75**, 3788 (1995).

[11] A. Imamoglu, D. D. Awschalom, G. Burkard, D. P. DiVincenzo, D. Loss, M. Sherwin, and A. Small, *Quantum information processing using quantum dot spins and cavity QED*, Phys. Rev. Lett. **83**, 4204 (1999).

[12] C. Monroe, D. Leibfried, B. E. King, D. M. Meekhof, W. M. Itano, and D. J. Wineland, *Simplified quantum logic with trapped ions*, Phys. Rev. A **55**, R2498 (1997).

[13] B. E. King, C. S. Wood, C. J. Myatt, Q. A. Turchette, D. Leibfried, W. M. Itano, C. Monroe, and D. J. Wineland, *Cooling the collective motion of trapped ions to initialze a quantum register*, Phys. Rev. Lett **81**, 1525 (1998).

[14] Note that σ^+ corresponds to a transition from $|1\rangle$ to $|0\rangle$, which accounts for the convention that $\sigma^+ a$ is a blue transition.

[15] M. H. Levitt, *Composite pulses*, Prog. NMR Spect. **18**, 61 (1986).